2.3. a.

$A \cdot x = y$

$$\begin{pmatrix} — 1 — \\ — 2 — \\ \vdots \\ — n — \end{pmatrix} \cdot \begin{pmatrix} | \\ x \\ | \end{pmatrix} = \begin{pmatrix} (— 1 —) \cdot \begin{pmatrix} | \\ x \\ | \end{pmatrix} \\ (— 2 —) \cdot \begin{pmatrix} | \\ x \\ | \end{pmatrix} \\ \vdots \\ (— n —) \underset{\uparrow}{\cdot} \begin{pmatrix} | \\ x \\ | \end{pmatrix} \end{pmatrix}$$

dot product

$$\Rightarrow \quad y_1 = 2x_1 + a_1 x_2$$

$$y_2 = 2x_2 + a_2 x_3$$

$$\boxed{\text{for } i = 3, \dots, n-1} \left\{ \begin{array}{l} y_3 = b_1 x_1 + 2x_3 + a_3 x_4 \\ y_4 = b_2 \cdot x_2 + 2x_4 + a_4 x_5 \\ \vdots \\ y_i = b_{i-2} x_{i-2} + 2 \cdot x_i + a_i \cdot x_{i+1} \\ \vdots \end{array} \right.$$

$$y_n = b_{n-2} \cdot x_{n-2} + 2 \cdot x_n .$$

A 2.3. c.

$$r_n = 2x_n \iff x_n = {r_n}/{2}$$

$$\left\{ \begin{array}{l} r_{n-1} = 2x_{n-1} + a_{n-1} \cdot x_n \iff x_{n-1} = \frac{1}{2} \cdot (r_{n-1} - a_{n-1} \cdot x_n) \\ \vdots \\ r_i = 2x_i + a_i \cdot x_{i+1} \iff x_i = \frac{1}{2} \cdot (r_i - a_i \cdot x_{i+1}) \end{array} \right.$$

$$\boxed{\text{for } i = n-1, \dots, 1}$$

2.3. d.

Basic Algorithm for Gaussian Elimination:
check Section 2.3.1 in the script!

$$A = \begin{pmatrix} \overset{\text{Pivot}}{\boxed{2}} & a_1 & 0 & 0 & \cdots \\ 0 & 2 & a_2 & 0 & \cdots \\ b_1 & 0 & 2 & a_3 & \cdots \\ \vdots & & & & \end{pmatrix}$$

we define:

$$l_{31} := a_{31}/a_{11} = {b_1}/{2}$$

$$\Rightarrow \overbrace{A_{3.}}^{\text{3rd row of A}} = A_{3.} - l_{31} \cdot A_{1.}$$

$\boxed{\text{Step 1}}$

$$\Rightarrow A = \begin{pmatrix} \overset{d_1}{\textcircled{2}} & a_1 & 0 & 0 & \cdots \\ \overset{c_1}{\boxed{0}} & \overset{d_2}{\textcircled{2}} & a_2 & 0 & \cdots \\ 0 & \overset{c_2}{\boxed{-\frac{b_1}{2} a_1}} & 2 & a_3 & \cdots \\ 0 & b_2 & 0 & 2 & \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & a_1 & 0 & 0 & 0 & \cdots \\ 0 & \overset{\text{pivot}\;\; d_2}{\boxed{\textcircled{2}}} & a_2 & 0 & 0 & \cdots \\ 0 & \overset{c_2}{\boxed{-\frac{b_1}{2} a_1}} & 2 & a_3 & 0 & \cdots \\ 0 & b_2 & 0 & 2 & a_4 & \cdots \end{pmatrix}$$

Step 2

$$l_{32} := a_{32} / a_{22} = c_2 / d_2$$

$$\Rightarrow A_{3.} = A_{3.} - l_{32} \cdot A_{2.} \quad , \quad r_3 = r_3 - l_{32} \cdot r_2$$

$$l_{42} := a_{42} / a_{22} = b_2 / d_2$$

$$\Rightarrow A_{4.} = A_{4.} - l_{42} \cdot A_{2.} \quad , \quad r_4 = r_4 - l_{42} \cdot r_2$$

$$\Rightarrow A = \begin{pmatrix} 2 & a_1 & 0 & 0 & 0 \\ 0 & 2 & a_2 & 0 & 0 \\ 0 & 0 & \overset{d_3}{\boxed{2 - \frac{c_2}{d_2} \cdot a_2}} & a_3 & 0 & \cdots \\ 0 & 0 & \underset{c_3}{\boxed{-\frac{b_2}{d_2} \cdot a_2}} & 2 & a_4 & \cdots \end{pmatrix}$$

and so on!

2.5.c

$ev_{new} = M^{-1} ev$ with

$$M = diag(d) + \underbrace{ev \cdot ev^T}_{\text{Rank-1-modification}}$$

$$\Rightarrow ev_{new} = diag(d)^{-1} ev - \frac{diag(d)^{-1} ev(ev^T (diag(d)^{-1} ev))}{1 + ev^T (diag(d)^{-1} ev)}$$

$d :=$ diag(d), use same denominator

$$\Rightarrow ev_{new} = \frac{d^{-1} ev (1 + ev^T (d^{-1} ev)) - d^{-1} ev (ev^T (d^{-1} ev))}{1 + ev^T (d^{-1} ev)}$$

$$= \frac{d^{-1} ev (1 + ev^T (d^{-1} ev) - ev^T (d^{-1} ev))}{1 + ev^T (d^{-1} ev)}$$

$$\Rightarrow ev_{new} = \underline{\underline{\frac{d^{-1} ev}{1 + ev^T (d^{-1} ev)}}} \quad \text{with } d^{-1} = \begin{pmatrix} 1/d_{11} & & 0 \\ & \ddots & \\ 0 & & 1/d_{nn} \end{pmatrix}$$

Further:

$$l_{new} = ev_{new}^T M ev_{new} = ev_{new}^T (diag(d) + ev \cdot ev^T) ev_{new}$$

$$= ev_{new}^T \cdot diag(d) \cdot ev_{new} + ev_{new}^T \cdot ev \cdot ev^T \cdot ev_{new}$$

(commutative) because real vect.

$$= ev_{new}^T diag(d) ev_{new} + ev_{new}^T \cdot ev \cdot ev_{new}^T \cdot ev$$

$$= \underline{\underline{ev_{new}^T diag(d) ev_{new} + (ev_{new}^T ev)^2}}$$

2.6. c.

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} \quad A^T = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}$$

$$A \cdot X = \begin{pmatrix} 2x_1 + x_2 & 2x_3 + x_4 \\ -x_1 + 3x_2 & -x_3 + 3x_4 \end{pmatrix}$$

$$X \cdot A^T = \begin{pmatrix} 2x_1 + x_3 & 3x_3 - x_1 \\ 2x_2 + x_4 & 3x_4 - x_2 \end{pmatrix}$$

$$\Rightarrow A \cdot X + X \cdot A^T = \begin{pmatrix} 4x_1 + x_2 + x_3 & -x_1 + 5x_3 + x_4 \\ -x_1 + 5x_2 + x_4 & -x_2 - x_3 + 6x_4 \end{pmatrix}$$

$\Rightarrow$ Find C, such that $\operatorname{vec}(A \cdot X + X \cdot A^T) = C \cdot \operatorname{vec}(X)$

$$\Rightarrow C \cdot \operatorname{vec}(X) = \begin{pmatrix} 4x_1 + x_2 + x_3 \\ -x_1 + 5x_2 + x_4 \\ -x_1 + 5x_3 + x_4 \\ -x_2 - x_3 + 6x_4 \end{pmatrix}$$ with $\operatorname{vec}(X) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

We find:

$$\Rightarrow C = \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \end{array}$$
$$C = \left(\begin{array}{cccc} 4 & 1 & 1 & 0 \\ -1 & 5 & 0 & 1 \\ -1 & 0 & 5 & 1 \\ 0 & -1 & -1 & 6 \end{array}\right).$$

Further, $\operatorname{vec}(A \cdot X + X \cdot A^T) = \operatorname{vec}(I)$ and

$$C \cdot \operatorname{vec}(X) = \operatorname{vec}(AX + XA^T) = \operatorname{vec}(I)$$

implies

$$b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \operatorname{vec}(I)$$

2.6.d

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}$$

$$A \cdot X = \begin{pmatrix} a_{11} x_1 + a_{12} x_2 & a_{11} x_3 + a_{12} x_4 \\ a_{21} x_1 + a_{21} x_2 & a_{21} \cdot x_3 + a_{22} x_4 . \end{pmatrix}$$

$$X \cdot A^T = \begin{pmatrix} x_1 a_{11} + x_3 a_{12} & x_1 a_{21} + x_3 a_{22} \\ x_2 a_{11} + x_4 a_{12} & x_2 \cdot a_{21} + x_4 \cdot a_{22} \end{pmatrix}$$

$$\mathrm{vec}(AX + XA^T) = \begin{pmatrix} a_{11} x_1 + a_{11} x_1 + a_{12} x_2 + a_{12} x_3 \\ a_{21} x_1 + a_{22} \cdot x_2 + a_{11} x_2 + \cancel{x_4} \, a_{12} x_4 \\ a_{21} \cdot x_1 + a_{11} \cdot x_3 + a_{22} \cdot x_3 + a_{12} x_4 \\ a_{21} x_2 + a_{21} x_3 + a_{22} \cdot x_4 + a_{22} \cdot x_4 \end{pmatrix}$$

$$\Rightarrow \quad \operatorname{vec}(AX + XA^T) = C \cdot \operatorname{vec}(X)$$

$$\Rightarrow \quad C = \left(\begin{array}{c|c} \overbrace{\begin{matrix} a_{11}+a_{11} & a_{12} \\ a_{21} & a_{22}+a_{11} \end{matrix}}^{A + a_{11}\cdot I} & \overbrace{\begin{matrix} a_{12} & 0 \\ 0 & a_{12} \end{matrix}}^{a_{12}\cdot I} \\ \hline \underbrace{\begin{matrix} a_{21} & 0 \\ 0 & a_{21} \end{matrix}}_{a_{21}\cdot I} & \underbrace{\begin{matrix} a_{11}+a_{22} & a_{12} \\ a_{21} & a_{22}+a_{22} \end{matrix}}_{A + a_{22}\cdot I} \end{array}\right)$$

$$= \left(\begin{array}{c|c} A + a_{11}\cdot I & a_{12}\cdot I \\ \hline a_{21}\cdot I & A + a_{22}\cdot I \end{array}\right)$$

$$= \begin{pmatrix} A & \underline{0} \\ \underline{0} & A \end{pmatrix} + \begin{pmatrix} a_{11} I & a_{12} I \\ a_{21} I & a_{22}\cdot I \end{pmatrix}$$

$$= I \otimes A + A \otimes I$$

$$\Rightarrow \quad \underline{\underline{C = I \otimes A + A \otimes I}}$$