4.3.b

NumCSE16
14.11.16
Ex. 4.3, 4.4
G. Accaputo

$u \in \mathcal{P}_{m-1}, \ v \in \mathcal{P}_{n-1}$

$u \cdot v \in \mathcal{P}_{m+n-2}$

$\dim \mathcal{P}_{m+n-2} = m+n-1$

coeff. of polynom

$$(u \cdot v)(x) = \sum_{j=0}^{m+n-2} \omega_j \cdot x^j \ , \ \omega_j = \sum_{l=0}^{j} v_l u_{j-l}$$

$j = 0, \ldots, m+n-2$

$\Rightarrow [\omega_0, \omega_1, \ldots, \omega_{2n-2}]^T = u * v$

component-wise multiplication

$\Rightarrow$ (convolution theorem): $u * v = F_n^{-1}((F_n u) \cdot (F_n v))$

Fourier matrix.

$\Rightarrow$ (direct convolution)

coeff of polynom u.

~~pad~~ vector $u_{new} = [u, 0, \ldots, 0]^T$

$n-1$ times (degree of v)

vector $v_{new} = [v, 0, \ldots, 0]^T$

$m-1$ times (degree of u).

coeff. of u·v polynom!

$\Rightarrow k = u_{new} * v_{new} = F_n^{-1}[(F_n u_{new}) \cdot (F_n v_{new})]$

4.3. f.

$$u * v = \mathcal{F}_n^{-1}\left((\mathcal{F}_n u)(\mathcal{F}_n v)\right)$$

$$\Rightarrow k = u * v$$

$$\Rightarrow (\hat{\cdot} = \mathcal{F}_n(\cdot)) \quad \hat{k} = \hat{u} \bullet \hat{v}$$

Fourier-transform of k

$$\Leftrightarrow \hat{v} = \frac{\hat{k}}{\hat{u}}$$

$$\Leftrightarrow \mathcal{F}_n v = (\mathcal{F}_n k) / (\mathcal{F}_n(u))$$

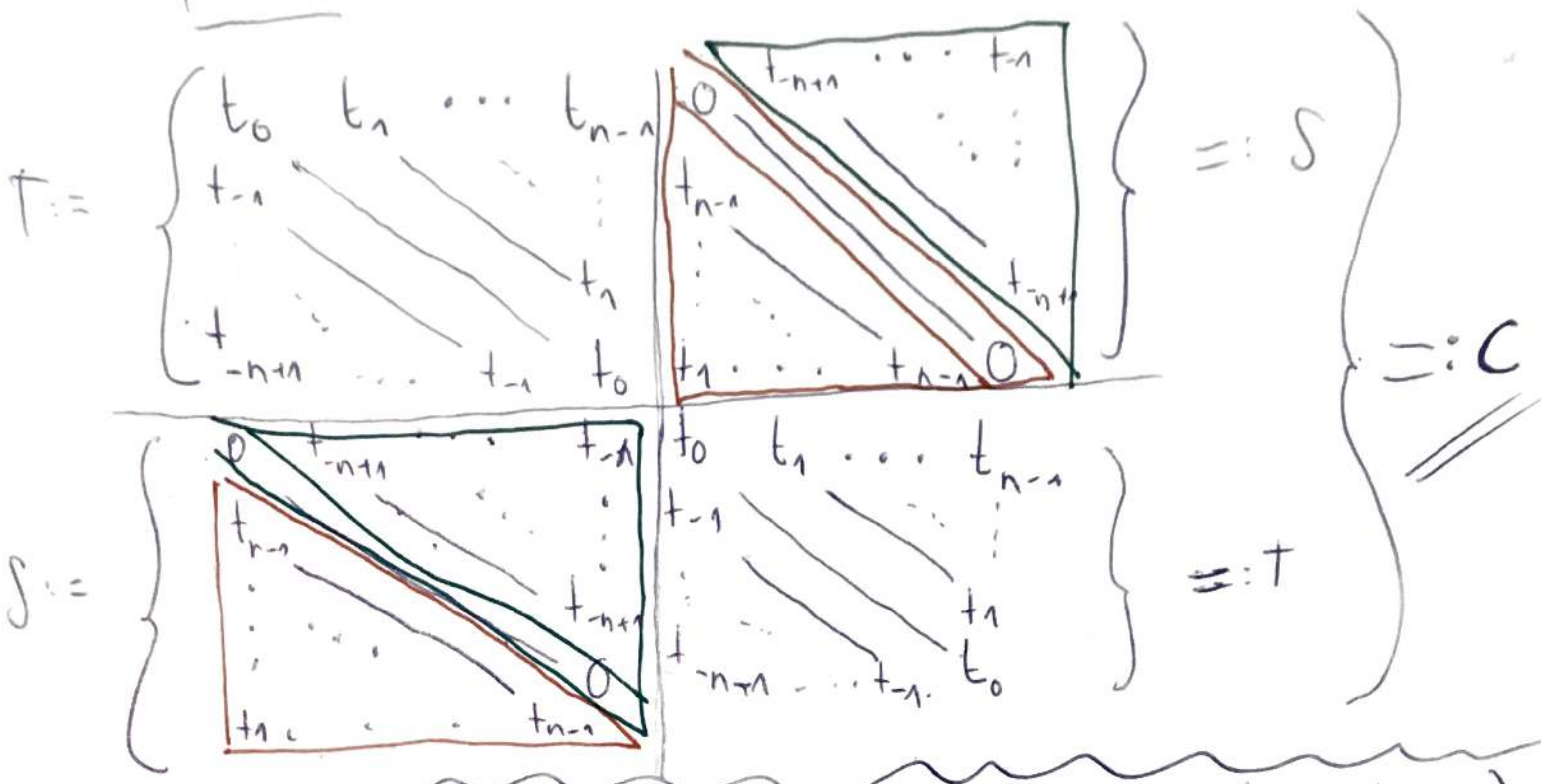
component-wise division!

$$\Leftrightarrow v = \mathcal{F}_n^{-1}\left[(\mathcal{F}_n k) / (\mathcal{F}_n u)\right]$$

(Deconvolution).

coeff. of polynom u·v!

4.4.a



4.4.b

$$(T)_{ij} = \begin{cases} (c)_{i-j+1}, & i \geq j \text{ (lower triang.)} \\ (r)_{j-i+1} & i < j \text{ (upper triang.)} \end{cases}$$

$$T = \begin{pmatrix} c_1 & r_2 & \cdots & r_n \\ c_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & r_2 \\ c_n & \cdots & c_2 & c_1 \end{pmatrix}$$ with $c_1 \overset{!}{=} r_1$ (diagonal!).

$$\Rightarrow C = \begin{bmatrix} T & S \\ S & T \end{bmatrix}$$ is circulant with

$c = [c_0, c_1, \dots, c_{2n-1}]^T$

$$C = \begin{bmatrix} c_0 & c_{2n-1} & \cdots & c_1 \\ c_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{2n-1} \\ c_{2n-1} & \cdots & c_1 & c_0 \end{bmatrix}$$

toepmult

$$\Rightarrow C \cdot x = y \Rightarrow y_k = \sum_{j=0}^{2n-1} c_{k,j} \cdot x_j \Rightarrow y = c * x$$

with $c = [c_0, \dots, c_{2n-1}]$

(see 4.2.17 in script!)

$$\Rightarrow C = \begin{bmatrix} T & S \\ S & T \end{bmatrix} = \begin{bmatrix} t_0 & t_1 & \cdots \\ t_{-1} & \ddots & \\ \vdots & & \\ t_{-n+1} & & \\ 0 & & \\ t_{n-1} & & \\ \vdots & & \\ t_1 & & \end{bmatrix}$$

$$c = [t_0, t_{-1}, \dots, t_{-n+1}, 0, t_{n-1}, \dots, t_1]^T$$

since $$T = \begin{bmatrix} t_0 & t_1 & \cdots & t_{n-1} \\ t_{-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_1 \\ t_{-n+1} & \cdots & t_{-1} & t_0 \end{bmatrix} = \begin{bmatrix} c_1 & r_2 & \cdots & r_n \\ c_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & r_2 \\ c_n & \cdots & c_2 & c_1 \end{bmatrix}$$

define $c = [c_1, c_2, \dots, c_n, 0, r_n, \dots, r_2]^T$

$= cr\_tmp$ in code 4.0.21 !

$$\Rightarrow C\tilde{x} = \begin{bmatrix} T & S \\ S & T \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} Tx \\ Sx \end{bmatrix} =: \tilde{y}$$

This is our result. $y = Tx$.

$$\Rightarrow C\tilde{x} = cr\_tmp * \tilde{x} = \tilde{y}$$

$$\Rightarrow Tx = y \text{ in } O(n \log n) \,!$$

since $cr\_tmp * \tilde{x} = \mathcal{F}_n^{-1}[(\mathcal{F}_n\, cr\_tmp)(\mathcal{F}_n \tilde{x})]$

and DFT in $O(n \log n)$.